

Armendariz Modules over Skew PBW Extensions

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Abstract

The aim of this paper is to develop the theory of skew Armendariz and quasi-Armendariz modules over skew PBW extensions. We generalize the results of several works in the literature concerning Ore extensions to another non-commutative rings which can not be expressed as iterated Ore extensions. As a consequence of our treatment, we extend and unify different results about the Armendariz, Baer, p.p., and p.q.-Baer properties for Ore extensions and skew PBW extensions.

Key words and phrases. Armendariz, Baer, quasi-Baer, p.p. and p.q.-Baer rings, skew PBW extensions.

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1 Introduction

In [23], Kaplansky defined a ring B as a *Baer* (*quasi-Baer*, which was defined by Clark [14]) ring, if the right annihilator of every nonempty subset (ideal) of B is generated by an idempotent (the objective of these rings is to abstract various properties of von Neumann algebras and complete \ast -regular rings; Clark used the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semi-group algebra). Another generalization of Baer rings are the p.p.-rings. A ring B is called *right* (*left*) *principally projective* (*p.p.* for short), if the right (*left*) annihilator of each element of B is generated by an idempotent (or equivalently, rings in which each principal right (*left*) ideal is projective). Birkenmeier et al. [12] defined a ring to be called a *right* (*left*) *principally quasi-Baer* (or simply *right* (*left*) *p.q.-Baer*) ring, if the right annihilator of each principal right (*left*) ideal of B is generated by an idempotent. Note that in a reduced ring B , B is Baer (p.p.) if and only if B is quasi-Baer (p.q.-Baer).

Commutative and noncommutative Baer, quasi-Baer, p.p. and p.q.-Baer rings have been investigated in the literature. For instance, in [5], Armendariz established the following proposition: if B is a reduced ring, then $B[x]$ is a Baer ring if and only if B is a Baer ring ([5], Theorem B). In fact, Armendariz showed an example to illustrate that the condition to be reduced is not superfluous. Birkenmeier et. al., in [12] showed that the quasi-Baer condition is preserved by many

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polynomial extensions, and in [10], they proved that a ring B is right p.q.-Baer if and only if $B[x]$ is right p.q.-Baer. In the context of Ore extensions (defined by Ore in [35]) given by $B[x; \sigma, \delta]$ with σ injective (also known as Ore extensions of injective type), we found several works in the literature, see [14], [10], [18], [12], [11], [17], [16], and others (in [42], [46], or [47] a detailed list of references is presented). Some of these works consider the case $\delta = 0$ and σ an automorphism, or the case where σ is the identity. It is important to say that the Baerness and quasi-Baerness of a ring B and an Ore extension $B[x; \sigma, \delta]$ of B does not depend on each other. More exactly, there are examples which show that there exists a Baer ring B but the Ore extension $B[x; \sigma, \delta]$ is not right p.q.-Baer; similarly, there exist Ore extensions $B[x; \sigma, \delta]$ which are quasi-Baer, but B is not quasi-Baer (see [18], Examples 8, 9 and 10 for more details).

With respect to the context of modules, Lee and Zhou in [26] introduced the notions of Baer, quasi-Baer and p.p.-modules in the following way: for a ring B and a right B -module M_B , (i) M_B is called *Baer (quasi-Baer)* if, for any subset (submodule) X of M , $\text{ann}_B(X) = eB$, where $e^2 = e \in B$; (ii) M_B is called *principally projective (p.p. for short) module (principally quasi-Baer module)* if, for any element $m \in M$, $\text{ann}_B(m) = eB$ ($\text{ann}_B(mB) = eB$), where $e^2 = e \in B$. It is important to remark that all these notions coincide with the ring definitions above, considering a ring B as a right module over itself. In other words, a ring B is Baer (quasi-Baer or p.p.) if and only if B_B is a Baer (quasi-Baer or p.p.) module. In fact, if B is a Baer (quasi-Baer or p.p.) ring, then for any right ideal I of B , I_B is a Baer (quasi-Baer or p.p.) module. Note that B is a right p.q.-Baer ring if and only if B_B is a p.q.-Baer module, and every submodule of a p.q.-Baer module is p.q.-Baer, and every Baer module is quasi-Baer.

Since the notion of reduced ring (a ring B is called *reduced* if it has no nonzero nilpotent elements; note that every reduced ring is abelian, i.e., every idempotent is central) is very important for characterizing the properties of being p.p. and p.q.-Baer (in [18], Lemma 1 it was proved that for a reduced ring B , B is a right p.p.-ring $\Leftrightarrow B$ is a p.p.-ring $\Leftrightarrow B$ is a right p.q.-Baer ring $\Leftrightarrow B$ is a p.q.-Baer ring), it is of interest to know its corresponding notion for the context of modules: M_B is called *reduced* (Lee and Zhou [26]), if for any elements $m \in M$, $a \in B$, $ma = 0$ implies $mB \cap Ma = 0$. Precisely, Lee and Zhou generalized several results of reduced rings to reduced modules.

The notion of Armendariz ring, which is the primary object of study in this paper, it has also been investigated. Let us recall briefly. In commutative algebra, a ring B is called *Armendariz* (the term was introduced by Rege and Chhawchharia in [36]) if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in B[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$, for every i, j . The interest of this notion lies in its natural and its useful role in understanding the relation between the annihilators of the ring B and the annihilators of the polynomial ring $B[x]$. In [5], Lemma 1, Armendariz showed that a reduced ring always satisfies this condition. Now, in the context of Ore extensions, Armendariz property has also been studied. For instance, Hirano in [17] defined a ring B to be *quasi-Armendariz* if whenever two polynomials $f(x) = \sum_{i=0}^m a_ix^i$, $g(x) = \sum_{j=0}^t b_jx^j \in B[x]$ satisfy $f(x)B[x]g(x) = 0$, then $a_iRb_j = 0$, for every i, j . In [19], Hong et. al., extended the Armendariz property of rings to skew polynomial rings $B[x; \alpha]$ with zero derivation. For an endomorphism α of a ring B , B is called an α -skew Armendariz ring,

if for polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ in $B[x; \alpha]$, $f(x)g(x) = 0$ implies $a_i\alpha^i(b_j) = 0$, for every $0 \leq i \leq n$, and $0 \leq j \leq m$. A more general treatment for the notion of Armendariz for Ore extensions with δ not necessarily zero, it was established by Nasr-Isfahani and Moussavi [34] using the notion of skew-Armendariz ring. It is important to say that the relations between Armendariz rings and Baer (quasi-Baer) rings have been also investigated in different papers, see for example [5], [36], [3], [10], [18], [12], [17], [19], [30], and others (see [42], [46], or [47] for a detailed list of references).

The notion of Armendariz for modules over Ore extensions also have been formulated. In [50], Zhang and Chen introduced the notion of α -skew Armendariz modules over Ore extensions with zero derivation ($\delta = 0$) in the following way: an B -module M is called α -skew Armendariz, if for polynomials $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[X]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in B[x; \alpha]$, $m(x)f(x) = 0$ implies $m_i\alpha^i(b_j) = 0$, for every $0 \leq i \leq k$ and $0 \leq j \leq n$ (Baser in [8] studied the relations between the set of annihilators in M_B and the set of annihilators in $M[X]$). A module M_B is called α -Armendariz, if M_B is α -compatible and α -skew-Armendariz ([26]). These authors also proved that B is an α -skew Armendariz ring if and only if every flat right B -module is α -skew Armendariz, and a module M_B is α -reduced, if M_B is α -compatible and reduced. A more general treatment about the notion of Armendariz module over Ore extensions with δ not necessarily zero it was presented by Alhevaz and Moussavi in [2]. There, they study the relationship between an B -module M_B and the general polynomial module $M[X]$ over the Ore extension $B[x; \alpha, \delta]$, and introduce the notions of skew-Armendariz modules and skew quasi-Armendariz modules which are generalizations of α -skew Armendariz modules [50] and α -reduced modules [26]. In fact, they also established several connections of the Baer, quasi-Baer and the p.p.-properties with the notion of skew Armendariz and skew quasi-Armendariz module. In this way, [2] extends and unifies several known results related to Armendariz rings and modules, such as [18], [19], [34], [50], and others, to general polynomial modules over Ore extensions.

With the aim of generalizing the results established about Armendariz and Baer properties in the mentioned papers above, in this article we are interested in a class of non-commutative rings of polynomial type more general than iterated Ore extensions (of injective type), the *skew Poincaré-Birkhoff-Witt extensions* (also known as σ -PBW extensions), where PBW denotes Poincaré-Birkhoff-Witt, introduced in [15] (see Examples 2.4 for a list of non-commutative rings which are σ -PBW extensions but not iterated Ore extensions). Actually, skew PBW extensions are more general than several families of non-commutative rings, such as universal enveloping algebras of finite dimensional Lie algebras, PBW extensions introduced by Bell and Goodearl in [9], almost normalizing extensions defined by McConnell and Robson in [31], solvable polynomial rings introduced by Kandri-Rody and Weispfenning in [22], and generalized by Kredel in [24], diffusion algebras studied by Isaev, Pyatov, and Rittenberg in [21], and other kind of non-commutative algebras of polynomial type. The importance of skew PBW extensions is that the coefficients do not necessarily commute with the variables, and these coefficients are not necessarily elements of fields (see Definition 2.1 below). In fact, the σ -PBW extensions contain well-known groups of algebras such as some types of G -algebras studied by Levandovskyy [29] and some PBW algebras defined by Bueso et. al., in [13] (both G -algebras and PBW algebras take coefficients in fields and assume that coefficients commute with variables), Auslander-Gorenstein rings, some Calabi-Yau

and skew Calabi-Yau algebras, some Artin-Schelter regular algebras, some Koszul algebras, quantum polynomials, some quantum universal enveloping algebras, and others (see [38], [28], and [48] for a detailed list of examples). For more details about the relation between σ -PBW extensions and another algebras with PBW bases, see [38] or [28].

Since Ore extensions of injective type are particular examples of σ -PBW extensions, and having in mind that several ring, module and homological properties of have been studied by the author and others for skew PBW extensions (see [15], [38], [39], [40], [41], [6], [27], [42], [7], [46], [47], [44], etc), we consider relevant to investigate the properties of Baer, quasi-Baer, p.p., p.q.-Baer, and Armendariz in the context of modules over these extensions (in [42], [43], [45], [46], and [47], these properties were investigated for σ -PBW extensions, for example, with the purpose of computing its Goldie dimension [40]) with the aim of establishing and generalizing several results in the literature for Ore extensions of injective type and σ -PBW extensions. In this way, our results generalizes several works concerning Ore extensions and σ -PBW extensions, such as [18], [17], [19], [26], [33], [34], [50], [2], [42], [45], [47], and [46]. We can say that the importance of our results is precisely to establish all these properties for those non-commutative rings which can not be expressed as Ore extensions.

The paper is organized as follows: In Section 2 we establish some useful results about σ -PBW extensions for the rest of the paper. Next, in Section 3 we introduce the notion of skew Armendariz and skew quasi-Armendariz modules based on [2]. First, Section 3.1 contains the definition of skew-Armendariz module for σ -PBW extensions, and in Section 3.2 we introduce the notion of skew quasi-Armendariz module. The more important results of this paper are presented in this section following the ideas established by Alhevaz and Moussavi in [2] for the case of Ore extensions. It is a remarkable fact that the tools employed in that paper are very useful for the study of Armendariz modules over rings which can not be expressed as Ore extensions. In this way, the techniques used here are fairly standard and follow the same path as other text on the subject. The results presented are new for skew PBW extensions and all they generalize others existing in the literature.

Throughout the paper, the word ring means a ring not necessarily commutative with unity, and all modules are right modules and $\text{ann}_B(X) := \{r \in B \mid Xr = 0\}$, where $X \subseteq B$, for any ring B .

2 Skew PBW extensions

In this section we establish some useful results about skew PBW extensions for the rest of the paper.

Definition 2.1 ([15], Definition 1). Let R and A be rings. We say that A is a σ -PBW extension (also known as *skew PBW extension*) of R , which is denoted by $A := \sigma(R)\langle x_1, \dots, x_n \rangle$, if the following conditions hold:

- (i) $R \subseteq A$;

- (ii) there exist elements $x_1, \dots, x_n \in A$ such that A is a left free R -module, with basis $\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$, and $x_1^0 \cdots x_n^0 := 1 \in \text{Mon}(A)$.
- (iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.
- (iv) For any elements $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$.

Proposition 2.2 ([15], Proposition 3). *Let A be a σ -PBW extension of R . For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \rightarrow R$ and an σ_i -derivation $\delta_i : R \rightarrow R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$. We write $\Sigma := \{\sigma_1, \dots, \sigma_n\}$, and $\Delta := \{\delta_1, \dots, \delta_n\}$.*

Definition 2.3 ([15], Definition 4). Let A be a σ -PBW extension of R .

- (a) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 2.1 are replaced by the following: (iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$; (iv') for any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$.
- (b) A is called *bijective*, if σ_i is bijective for each $1 \leq i \leq n$, and $c_{i,j}$ is invertible, for any $1 \leq i < j \leq n$.

Examples 2.4. If $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is an iterated Ore extension where

- σ_i is injective, for $1 \leq i \leq n$;
- $\sigma_i(r), \delta_i(r) \in R$, for every $r \in R$ and $1 \leq i \leq n$;
- $\sigma_j(x_i) = cx_i + d$, for $i < j$, and $c, d \in R$, where c has a left inverse;
- $\delta_j(x_i) \in R + Rx_1 + \cdots + Rx_n$, for $i < j$,

then $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] \cong \sigma(R)\langle x_1, \dots, x_n \rangle$ ([28], p. 1212). Note that σ -PBW extensions of endomorphism type are more general than iterated Ore extensions $R[x_1; \sigma_1] \cdots [x_n; \sigma_n]$, and in general, σ -PBW extensions are more general than Ore extensions of injective type.

Next, we present some non-commutative rings which are σ -PBW extensions but they can not be expressed as iterated Ore extensions (see [28] for the reference of every example).

- (a) Let k be a commutative ring and \mathfrak{g} a finite dimensional Lie algebra over k with basis $\{x_1, \dots, x_n\}$. The *universal enveloping algebra* of \mathfrak{g} , denoted $\mathcal{U}(\mathfrak{g})$, is a skew PBW extension of k , since $x_i r - r x_i = 0$, $x_i x_j - x_j x_i = [x_i, x_j] \in \mathfrak{g} = k + kx_1 + \cdots + kx_n$, $r \in k$, for $1 \leq i, j \leq n$. In particular, the *universal enveloping algebra of a Kac-Moody Lie algebra* is a skew PBW extension of a polynomial ring.
- (b) The *universal enveloping ring* $\mathcal{U}(V, R, \mathbb{k})$, where R is a \mathbb{k} -algebra, and V is a \mathbb{k} -vector space which is also a Lie ring containing R and \mathbb{k} as Lie ideals with suitable relations. The enveloping ring $\mathcal{U}(V, R, \mathbb{k})$ is a finite skew PBW extension of R if $\dim_{\mathbb{k}}(V/R)$ is finite.

- (c) Let $k, \mathfrak{g}, \{x_1, \dots, x_n\}$ and $\mathcal{U}(\mathfrak{g})$ be as in the previous example; let R be a k -algebra containing k . The *tensor product* $A := R \otimes_k \mathcal{U}(\mathfrak{g})$ is a skew PBW extension of R , and it is a particular case of *crossed product* $R * \mathcal{U}(\mathfrak{g})$ of R by $\mathcal{U}(\mathfrak{g})$, which is a skew PBW extension of R .
- (d) The *twisted or smash product differential operator ring* $R \#_\sigma \mathcal{U}(\mathfrak{g})$, where \mathfrak{g} is a finite-dimensional Lie algebra acting on R by derivations, and σ is Lie 2-cocycle with values in R .
- (e) Diffusion algebras arise in physics as a possible way to understand a large class of 1-dimensional stochastic process [21]. A *diffusion algebra* \mathcal{A} with parameters $a_{ij} \in \mathbb{C} \setminus \{0\}, 1 \leq i, j \leq n$, is an algebra over \mathbb{C} generated by variables x_1, \dots, x_n subject to relations $a_{ij}x_i x_j - b_{ij}x_j x_i = r_j x_i - r_i x_j$, whenever $i < j$, $b_{ij}, r_i \in \mathbb{C}$ for all $i < j$. \mathcal{A} admits a PBW-basis of standard monomials $x_1^{i_1} \cdots x_n^{i_n}$, that is, \mathcal{A} is a diffusion algebra if these standard monomials are a \mathbb{C} -vector space basis for \mathcal{A} . From Definition 2.1, (iii) and (iv), it is clear that the family of skew PBW extensions are more general than diffusion algebras. We will denote $q_{ij} := \frac{b_{ij}}{a_{ij}}$. The parameter q_{ij} can be a root of unity if and only if is equal to 1. It is therefore reasonable to assume that these parameters not to be a root of unity other than 1. If all coefficients q_{ij} are nonzero, then the corresponding diffusion algebra have a PBW basis of standard monomials $x_1^{i_1} \cdots x_n^{i_n}$, and hence these algebras are skew PBW extensions. More precisely, $\mathcal{A} \cong \sigma(\mathbb{C})\langle x_1, \dots, x_n \rangle$.

It is important to say that σ -PBW extensions contains various well-known groups of algebras such as PBW extensions [9], the almost normalizing extensions [31], solvable polynomial rings [22], and [24], diffusion algebras [21], some types of Auslander-Gorenstein rings, some skew Calabi-Yau algebras, some Artin-Schelter regular algebras, some Koszul algebras, quantum polynomials, some quantum universal enveloping algebras, etc. In comparison with G -algebras [29] or PBW algebras [13], σ -PBW extensions do not assume that the ring of coefficients is a field neither that the coefficients commute with the variables, so that skew PBW extensions are not included in these algebras. Indeed, the G -algebras with $d_{i,j}$ linear (recall that for these algebras $x_j x_i = c_{i,j} x_i x_j + d_{i,j}$, $1 \leq i < j \leq n$), are particular examples of σ -PBW extensions. A detailed list of examples of skew PBW extensions and its relations with another algebras with PBW bases is presented in [38] and [28].

Definition 2.5 ([15], Definition 6). Let A be a σ -PBW extension of R . Then:

- (i) for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.
- (ii) For $X = x^\alpha \in \text{Mon}(A)$, $\exp(X) := \alpha$, $\deg(X) := |\alpha|$, and $X_0 := 1$. The symbol \succeq will denote a total order defined on $\text{Mon}(A)$ (a total order on \mathbb{N}^n). For an element $x^\alpha \in \text{Mon}(A)$, $\exp(x^\alpha) := \alpha \in \mathbb{N}^n$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. Every element $f \in A$ can be expressed uniquely as $f = a_0 + a_1 X_1 + \cdots + a_m X_m$, with $a_i \in R$, and $X_m \succ \cdots \succ X_1$ (eventually, we will use expressions as $f = a_0 + a_1 Y_1 + \cdots + a_m Y_m$, with $a_i \in R$, and $Y_m \succ \cdots \succ Y_1$). With this notation, we define $\text{lm}(f) := X_m$, the *leading monomial* of f ; $\text{lc}(f) := a_m$, the *leading coefficient* of f ; $\text{lt}(f) := a_m X_m$, the *leading term* of f ; $\exp(f) := \exp(X_m)$, the *order* of f ; and $E(f) := \{\exp(X_i) \mid 1 \leq i \leq t\}$. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$. Finally, if $f = 0$, then $\text{lm}(0) := 0$, $\text{lc}(0) := 0$, $\text{lt}(0) := 0$. We

also consider $X \succ 0$ for any $X \in \text{Mon}(A)$. For a detailed description of monomial orders in skew PBW extensions, see [15], Section 3.

Proposition 2.6 ([15], Theorem 7). *If A is a polynomial ring with coefficients in R with respect to the set of indeterminates $\{x_1, \dots, x_n\}$, then A is a skew PBW extension of R if and only if the following conditions hold:*

- (i) *for each $x^\alpha \in \text{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$, or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α .*
- (ii) *For each $x^\alpha, x^\beta \in \text{Mon}(A)$, there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$, or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.*

Remark 2.7. About Proposition 2.6, we have two observations:

- (i) ([42], Proposition 2.9) If $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $r \in R$, then

$$\begin{aligned}
x^\alpha r &= x_1^{\alpha_1} x_2^{\alpha_2} \dots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} r = x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}} \left(\sum_{j=1}^{\alpha_n} x_n^{\alpha_n-j} \delta_n(\sigma_n^{j-1}(r)) x_n^{j-1} \right) \\
&+ x_1^{\alpha_1} \dots x_{n-2}^{\alpha_{n-2}} \left(\sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{\alpha_{n-1}-j} \delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(r))) x_{n-1}^{j-1} \right) x_n^{\alpha_n} \\
&+ x_1^{\alpha_1} \dots x_{n-3}^{\alpha_{n-3}} \left(\sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{\alpha_{n-2}-j} \delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n}(r)))) x_{n-2}^{j-1} \right) x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&+ \dots + x_1^{\alpha_1} \left(\sum_{j=1}^{\alpha_2} x_2^{\alpha_2-j} \delta_2(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\dots(\sigma_n^{\alpha_n}(r))))) x_2^{j-1} \right) x_3^{\alpha_3} x_4^{\alpha_4} \dots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&+ \sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\dots(\sigma_n^{\alpha_n}(r)))) x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad \sigma_j^0 := \text{id}_R \text{ for } 1 \leq j \leq n.
\end{aligned}$$

- (ii) ([42], Remark 2.10) Using (i), it follows that for the product $a_i X_i b_j Y_j$, if $X_i := x_1^{\alpha_{i1}} \dots x_n^{\alpha_{in}}$ and $Y_j := x_1^{\beta_{j1}} \dots x_n^{\beta_{jn}}$, then

$$\begin{aligned}
a_i X_i b_j Y_j &= a_i \sigma^{\alpha_i}(b_j) x^{\alpha_i} x^{\beta_j} + a_i p_{\alpha_{i1}, \sigma_{i2}^{\alpha_{i2}}(\dots(\sigma_{in}^{\alpha_{in}}(b)))} x_2^{\alpha_{i2}} \dots x_n^{\alpha_{in}} x^{\beta_j} \\
&+ a_i x_1^{\alpha_{i1}} p_{\alpha_{i2}, \sigma_3^{\alpha_{i3}}(\dots(\sigma_{in}^{\alpha_{in}}(b)))} x_3^{\alpha_{i3}} \dots x_n^{\alpha_{in}} x^{\beta_j} \\
&+ a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} p_{\alpha_{i3}, \sigma_{i4}^{\alpha_{i4}}(\dots(\sigma_{in}^{\alpha_{in}}(b)))} x_4^{\alpha_{i4}} \dots x_n^{\alpha_{in}} x^{\beta_j} \\
&+ \dots + a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \dots x_{i(n-2)}^{\alpha_{i(n-2)}} p_{\alpha_{i(n-1)}, \sigma_{in}^{\alpha_{in}}(b)} x_n^{\alpha_{in}} x^{\beta_j} \\
&+ a_i x_1^{\alpha_{i1}} \dots x_{i(n-1)}^{\alpha_{i(n-1)}} p_{\alpha_{in}, b} x^{\beta_j}.
\end{aligned}$$

In this way, when we compute every summand of $a_i X_i b_j Y_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ 's and δ 's depending of the coordinates of α_i .

3 Armendariz modules over σ -PBW extensions

In this section we introduce the notions of skew Armendariz module and skew quasi-Armendariz module over σ -PBW extensions. We start defining the modules which we are going to study.

From Definition 2.1 we know that if A is a σ -PBW extension of a ring R , then A is a left free R -module. Now, Remark 2.7 (i) says us how to multiply elements of R with elements of $\text{Mon}(A)$, so that if we consider a right R -module M_R , we can consider the polynomial module $M\langle X \rangle_A$ over A . More precisely, as a set, the elements of $M\langle X \rangle_A$ are of the form $m_0 + m_1X_1 + \cdots + m_tX_t$, $m_i \in M_R$ and $X_i \in \text{Mon}(A)$, for every i . If $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $r \in R$, then the action of A on these elements follow the rule established Remark 2.7 (ii). This fact is precisely because it suffices to define the action of monomials of A on monomials in $M\langle X \rangle_A$. In other words, if $m_i x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$ and $b_j x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}}$ are elements of $M\langle X \rangle$ and A , respectively, then we multiply these both elements following the rule

$$\begin{aligned}
m_i x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} b_j x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}} &= m_i \sigma^{\alpha_i}(b_j) x^{\alpha_i} x^{\beta_j} + m_i p_{\alpha_{i1}, \sigma_{i2}^{\alpha_{i2}}(\dots(\sigma_{in}^{\alpha_{in}}(b)))} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
&\quad + m_i x_1^{\alpha_{i1}} p_{\alpha_{i2}, \sigma_3^{\alpha_{i3}}(\dots(\sigma_{in}^{\alpha_{in}}(b)))} x_3^{\alpha_{i3}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
&\quad + m_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} p_{\alpha_{i3}, \sigma_{i4}^{\alpha_{i4}}(\dots(\sigma_{in}^{\alpha_{in}}(b)))} x_4^{\alpha_{i4}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
&\quad + \cdots + m_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_{i(n-2)}^{\alpha_{i(n-2)}} p_{\alpha_{i(n-1)}, \sigma_{in}^{\alpha_{in}}(b)} x_n^{\alpha_{in}} x^{\beta_j} \\
&\quad + m_i x_1^{\alpha_{i1}} \cdots x_{i(n-1)}^{\alpha_{i(n-1)}} p_{\alpha_{in}, b} x^{\beta_j}.
\end{aligned} \tag{3.1}$$

This guarantees that $M\langle X \rangle$ is really an A -module. In this way, when we compute every summand of $m_i x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} b_j x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}}$, we obtain products of the coefficient m_i with several evaluations of b_j in σ 's and δ 's, depending of the coordinates of α_i .

The purpose in the next two sections is to study the existing relations between an R -module M_R and the polynomial module $M\langle X \rangle$ over the skew-PBW extension A of R . Therefore, we extend the notions of skew-Armendariz modules and skew quasi-Armendariz modules introduced by Alhevaz and Moussavi [2] for the case of Ore extensions, and hence we generalize the concepts of α -skew Armendariz modules [50] and α -reduced modules [26].

3.1 Skew-Armendariz modules

In this section we introduce the notion of skew-Armendariz module for σ -PBW extensions. As we said above, our treatment generalize [26], [50], and [2].

Let us briefly recall some definitions about the notion of Armendariz for modules: (i) ([50], Definition 2.1); let B be a ring with an endomorphism α and M_B an B -module. M_B is called an α -skew Armendariz module, if for polynomials $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[X]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in B[x; \alpha]$, $m(x)f(x) = 0$ implies $m_i\alpha^i(b_j) = 0$, for every $0 \leq i \leq k$ and $0 \leq j \leq n$. (ii) ([2], Definition 2.2); let B be a ring with an endomorphism α and α -derivation δ . Let M_B be an B -module. M_B is an (α, δ) -skew Armendariz module, if for polynomials $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[X]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in B[x; \alpha, \delta]$,

$m(x)f(x) = 0$ implies $m_i x^i b_j x^j = 0$, for every $0 \leq i \leq k$ and $0 \leq j \leq n$ (it is clear that if $\delta = 0$, this definition coincides with the definition of α -skew Armendariz module). (iii) ([2], Definition 2.3); let B be a ring with an endomorphism α and a α -derivation δ . A right B -module M_B is called *skew-Armendariz module*, if for polynomials $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[X]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in B[x; \alpha, \delta]$, the equality $m(x)f(x) = 0$ implies $m_0b_j = 0$, for every $0 \leq j \leq n$. All these definitions are generalized in the next definition for the context of σ -PBW extensions.

Definition 3.1. Let A be a σ -PBW extension of a ring R , and let M_R be a right R -module. M_R is called a *skew-Armendariz module*, if for elements $m = m_0 + m_1X_1 + \cdots + m_kX_k \in M\langle X \rangle$ and $f = b_0 + b_1Y_1 + \cdots + b_tY_t \in A$ with $mf = 0$, we have $m_0b_j = 0$, for every $0 \leq j \leq t$.

Note that B is skew-Armendariz if B_B is a skew-Armendariz module. Now, since the notion of skew-Armendariz module is a generalization of an α -skew-Armendariz module ([2], Theorem 2.4), both concepts in the context of Ore extensions, and our notion of skew-Armendariz module in Definition 3.1 is formulated for σ -PBW extensions, which are more general than Ore extensions (with α injective), then our skew-Armendariz module notion is more general than α -skew-Armendariz module. Nevertheless, we can establish the following result without proof. Theorem 3.2 generalizes [2], Theorem 2.4, Corollaries 2.5, 2.6 and 2.7.

Theorem 3.2. *If A is a quasi-commutative skew PBW extension of a ring R , and M_R is a right module, then M_R is Σ -skew Armendariz if and only if for every polynomials $m = m_0 + m_1X + \cdots + m_kX_k \in M\langle X \rangle$, and $f = b_0 + b_1X_1 + \cdots + b_mX_m \in A$, the equality $mf = 0$ implies $m_0b_j = 0$, for every $0 \leq j \leq m$.*

The next definition introduce a more general class of modules than those established in Definition 3.1.

Definition 3.3. Let A be a σ -PBW extension of a ring R , and let M_R be a right R -module. M_R is called a *linearly skew-Armendariz module*, if for linear polynomials $m = m_0 + m_1x + \cdots + m_nx_n \in M\langle X \rangle$ and $g(x) = b_0 + b_1x + \cdots + b_nx_n \in A$ with $mg = 0$, we have $m_0b_j = 0$, for every $0 \leq j \leq n$.

In [4], Annin introduce the notion of compatibility for modules in the following way: given a module M_B , an endomorphism $\alpha : B \rightarrow B$ and an α -derivation $\delta : R \rightarrow R$, M_B is α -compatible if for each $m \in M$ and $r \in B$, we have $mr = 0 \Leftrightarrow m\alpha(r) = 0$. Moreover, M_R is δ -compatible if for each $m \in M$ and $r \in B$, we have $mr = 0 \Rightarrow m\delta(r) = 0$. If M_B is both α -compatible and δ -compatible, M_B is called (α, δ) -compatible. In [46], Definition 3.2, the author defined the notion of compatibility for skew PBW extensions in the following way (this definition extends [16]): consider a ring R with a family of endomorphisms Σ and a family of Σ -derivations Δ (Proposition 2.2). (i) R is said to be Σ -compatible, if for each $a, b \in R$, $a\sigma^\alpha(b) = 0$ if and only if $ab = 0$, for every $\alpha \in \mathbb{N}^n$; (ii) R is said to be Δ -compatible, if for each $a, b \in R$, $ab = 0$ implies $a\delta^\beta(b) = 0$, for every $\beta \in \mathbb{N}^n$; (iii) if R is both Σ -compatible and Δ -compatible, R is called (Σ, Δ) -compatible. As it was established in [46], Proposition 3.3, the importance of (Σ, Δ) -compatible rings is that they are more general than Σ -rigid rings defined and characterized by the author in [42] in terms of the properties of being Baer, p.p., p.q., and p.q.-Baer (Σ -rigid rings are a generalization of α -rigid rings defined by Krempa in [25] and studied by Hong et. al., [18]). Next, we extend this definition of compatibility for the context of modules over skew PBW extensions.

Definition 3.4. If A is a σ -PBW extension of a ring R , and M_R is a right R -module, then M_R is called Σ -compatible, if for every $m \in M$ and $r \in R$, $mr = 0 \Leftrightarrow m\sigma^\alpha(r) = 0$, for any $\alpha \in \mathbb{N}^n$. M_R is called Δ -compatible, if for every $m \in M$ and $r \in R$, $mr = 0 \Rightarrow m\delta^\beta(r) = 0$, for any $\beta \in \mathbb{N}^n$. If M_R is both Σ -compatible and Δ -compatible, then M_R is called (Σ, Δ) -compatible.

From [4], Lemma 2.16, we know that in the case of Ore extensions, a module M_B is (α, δ) -compatible if and only if the polynomial extension $M\langle X \rangle_B$ is (α, δ) -compatible. This assertion is generalized in the next proposition.

Proposition 3.5. *If A is a σ -PBW extension of R , and M_R is a right R -module, then M_R is (Σ, Δ) -compatible if and only if $M\langle X \rangle_R$ is (Σ, Δ) -compatible.*

Proof. The proofs follow from the definitions. □

The following proposition is a direct consequence from [2], Lemma 2.14.

Proposition 3.6. *If M_R is a right R -module, then the following conditions are equivalent:*

- (i) M_R is reduced and (Σ, Δ) -compatible.
- (ii) for any $m \in M$ and $r \in R$, the following conditions hold:
 - (a) $mr = 0$ implies $mRr = 0$;
 - (b) $mr = 0$ implies $m\delta^\beta(r) = 0$, for any $\beta \in \mathbb{N}^n$;
 - (c) $mr = 0$ if and only if $m\sigma^\theta(r) = 0$, for any $\theta \in \mathbb{N}^n$;
 - (d) $mr^2 = 0$ implies $mr = 0$.

The following proposition generalizes [2], Lemma 2.15.

Proposition 3.7. *If M_R is an (Σ, Δ) -compatible module, and $m \in M$, $a, b \in R$, then we have the following assertions:*

- (i) if $ma = 0$, then $m\sigma^\theta(a) = 0 = m\delta^\theta(a)$, for any element $\theta \in \mathbb{N}^n$;
- (ii) if $mab = 0$, then $m\sigma_i(\delta^\theta(a))\delta_i(b) = m\sigma^\beta(\delta_i(a))\delta^\theta(b)$, and so, $ma\delta^\theta(b) = 0 = m\delta^\theta(a)b$, for any elements $\beta, \theta \in \mathbb{N}^n$, and $i = 1, \dots, n$;
- (iii) $\text{ann}_R(\{ma\}) = \text{ann}_R(m\sigma_i(a)) = \text{ann}_R(\{m\delta_i(a)\})$, for every $i = 1, \dots, n$.

Proposition 3.8 generalizes [2], Lemma 2.16, from Ore extensions to σ -PBW extensions.

Proposition 3.8. *If A is a σ -PBW extension of a ring R , M_R is a (Σ, Δ) -compatible right R -module, $m = m_0 + m_1X_1 + \dots + m_kX_k$ is an element of $M\langle X \rangle$, and $a \in B$, then $mr = 0$ if and only if $m_i r = 0$, for every $0 \leq i \leq k$.*

Proof. Suppose that $m_i r = 0$, for every $0 \leq i \leq k$. Since

$$\begin{aligned}
mr &= (m_0 + m_1X_1 + \dots + m_kX_k)r \\
&= m_0r + m_1X_1r + \dots + m_kX_kr \\
&= m_0r + m_1(\sigma^{\alpha_1}(r)X_1 + p_{\alpha_1, r}) + \dots + m_k(\sigma^{\alpha_k}(r)X_k + p_{\alpha_k, r}) \\
&= m_0r + m_1\sigma^{\alpha_1}(r)X_1 + m_1p_{\alpha_1, r} + \dots + m_k\sigma^{\alpha_k}(r)X_k + m_kp_{\alpha_k, r},
\end{aligned} \tag{3.2}$$

where $\alpha_i = \exp(X_i)$, $p_{\alpha_i, r} = 0$, or, $\deg(p_{\alpha_i, r}) < |\alpha_i|$, if $p_{\alpha_i, r} \neq 0$, for every i , and using the equality $m_i r = 0$ with the expression (3.1) and the (Σ, Δ) -compatibility of M_R , we conclude that $mr = 0$.

Now, suppose that $mr = 0$. From expression (3.2) we can see that $\text{lc}(mr) = m_k \sigma^{\alpha_k}(r)$, so by the Σ -compatibility of M_R , we obtain $m_k r = 0$. Hence, expression (3.1) and (Σ, Δ) -compatibility of M_R imply that $p_{\alpha_k, r} = 0$, so mr reduces to

$$mr = m_0 r + m_1 \sigma^{\alpha_1}(r) X_1 + m_1 p_{\alpha_1, r} + \cdots + m_{k-1} \sigma^{\alpha_{k-1}}(r) X_{k-1} + m_{k-1} p_{\alpha_{k-1}, r}.$$

Again, since $\text{lc}(mr) = m_{k-1} \sigma^{\alpha_{k-1}}(r) = 0$, from Σ -compatibility of M_R we can assert that $m_{k-1} r = 0$. In this way, expression (3.1) and (Σ, Δ) -compatibility of M_R imply that $p_{\alpha_{k-1}, r} = 0$, so mr takes the form

$$mr = m_0 r + m_1 \sigma^{\alpha_1}(r) X_1 + m_1 p_{\alpha_1, r} + \cdots + m_{k-2} \sigma^{\alpha_{k-2}}(r) X_{k-2} + m_{k-2} p_{\alpha_{k-2}, r}.$$

Continuing in this way we can show that $m_k r = m_{k-1} r = m_{k-2} r = \cdots = m_1 r = m_0 r$, which concludes the proof. \square

The next proposition generalizes [2], Proposition 2.17 and Corollary 2.18.

Proposition 3.9. *A module M_R is Σ -reduced if and only if the extension $M\langle X \rangle_R$ is an Σ -reduced module.*

Theorem 3.10 generalizes [2], Theorem 2.19.

Theorem 3.10. *If M_R is an (Σ, Δ) -compatible and reduced module, then M_R is skew-Armendariz.*

Proof. Consider the elements $m = m_0 + m_1 X_1 + \cdots + m_k X_k \in M\langle X \rangle$, $f = b_0 + b_1 Y_1 + \cdots + b_t Y_t \in A$, with $mf = 0$. We have $mf = (m_0 + m_1 X_1 + \cdots + m_k X_k)(b_0 + b_1 Y_1 + \cdots + b_t Y_t) = \sum_{l=0}^{k+t} \left(\sum_{i+j=l} m_i X_i b_j Y_j \right)$. Note that $\text{lc}(mf) = m_k \sigma^{\alpha_k}(b_t) c_{\alpha_k, \beta_t} = 0$. Since A is bijective, $m_k \sigma^{\alpha_k}(b_t) = 0$, and by the Σ -compatibility of M_R , $m_k b_t = 0$. The idea is to prove that $m_p b_q = 0$ for $p + q \geq 0$. We proceed by induction. Suppose that $m_p b_q = 0$ for $p + q = k + t, k + t - 1, k + t - 2, \dots, l + 1$ for some $l > 0$. By Proposition 3.6 and expression (3.1), we obtain $m_p X_p b_q Y_q = 0$, for these values of $p + q$. In this way, we only consider the sum of the products $m_u X_u b_v Y_v$, where $u + v = l, l - 1, l - 2, \dots, 0$. Fix u and v . Consider the sum of all terms of mf having exponent $\alpha_u + \beta_v$. From expression (3.1), Proposition 3.6, and the assumption $mf = 0$, we know that the sum of all coefficients of all these terms can be written as

$$m_u \sigma^{\alpha_u}(b_v) c_{\alpha_u, \beta_v} + \sum_{\alpha_{u'} + \beta_{v'} = \alpha_u + \beta_v} m_{u'} \sigma^{\alpha_{u'}}(\sigma' \text{'s and } \delta' \text{'s evaluated in } b_{v'}) c_{\alpha_{u'}, \beta_{v'}} = 0. \quad (3.3)$$

By assumption, we know that $m_p b_q = 0$, for $p + q = k + t, k + t - 1, \dots, l + 1$. So, Proposition 3.6 guarantees that the product

$$m_p(\sigma' \text{'s and } \delta' \text{'s evaluated in } b_q) \quad (\text{any order of } \sigma' \text{'s and } \delta' \text{'s})$$

is equal to zero. Then $[(\sigma' \text{'s and } \delta' \text{'s evaluated in } b_q) a_p]^2 = 0$, and hence we obtain the equality $(\sigma' \text{'s and } \delta' \text{'s evaluated in } b_q) m_p = 0$ (M_R is reduced). In this way, multiplying (3.5) by m_l , and using the fact that the elements $c_{i,j}$ in Definition 2.1 (iv) are in the center of R ,

$$m_u \sigma^{\alpha_u}(b_v) a_k c_{\alpha_u, \beta_v} + \sum_{\alpha_{u'} + \beta_{v'} = \alpha_u + \beta_v} m_{u'} \sigma^{\alpha_{u'}}(\sigma' \text{'s and } \delta' \text{'s evaluated in } b_{v'}) m_l c_{\alpha_{u'}, \beta_{v'}} = 0, \quad (3.4)$$

whence, $m_u \sigma^{\alpha_u}(b_0) a_l = 0$. Since $u + v = l$ and $v = 0$, then $u = l$, so $m_l \sigma^{\alpha_l}(b_0) m_l = 0$, i.e., $[m_l \sigma^{\alpha_l}(b_0)]^2 = 0$, from which $m_l \sigma^{\alpha_l}(b_0) = 0$ and $m_l b_0 = 0$, by Proposition 3.6. Therefore, we now have to study the expression (3.5) for $0 \leq u \leq l - 1$ and $u + v = l$. If we multiply (3.6) by m_{l-1} , we obtain

$$m_u \sigma^{\alpha_u}(b_v) m_{l-1} c_{\alpha_u, \beta_v} + \sum_{\alpha_{u'} + \beta_{v'} = \alpha_u + \beta_v} m_{u'} \sigma^{\alpha_{u'}}(\sigma' \text{'s and } \delta' \text{'s evaluated in } b_{v'}) m_{k-1} c_{\alpha_{u'}, \beta_{v'}} = 0.$$

Using a similar reasoning as above, we can see that $m_u \sigma^{\alpha_u}(b_1) m_{l-1} c_{\alpha_u, \beta_1} = 0$. Since A is bijective, $m_u \sigma^{\alpha_u}(b_1) m_{l-1} = 0$, and using the fact $u = l - 1$, we have $[m_{l-1} \sigma^{\alpha_{l-1}}(b_1)] = 0$, which imply $m_{l-1} \sigma^{\alpha_{l-1}}(b_1) = 0$, that is, $m_{l-1} b_1 = 0$. Continuing in this way we prove that $m_i b_j = 0$, for $i + j = l$. Therefore $a_i b_j = 0$, for $0 \leq i \leq m$ and $0 \leq j \leq t$, which concludes the proof. \square

Proposition 3.11 ([2], Proposition 2.22). *Suppose that M is a flat right B -module. Then for every exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$, where F is B -free, we have $FI \cap K = KI$ for each left ideal I of B . In particular, we have $Fa \cap K = Ka$ for each element a of B .*

Proposition 3.12 generalizes [2], Proposition 2.23.

Proposition 3.12. *If A is a σ -PBW extension of a ring R , then B is a skew-Armendariz ring if and only if every flat B -module M is skew-Armendariz.*

Proof. Consider M a flat R -module, and an exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$, where F is free over R . If b is an element of F , then $\bar{b} = b + K$ is an element of M . Let $f = \bar{b}_0 + \bar{b}_1 X_1 + \cdots + \bar{b}_k X_k \in M\langle X \rangle$ and $g = a_0 + a_1 Y_1 + \cdots + a_t Y_t \in A$ with $fg = 0$. Let us prove that $\bar{b}_0 a_j = 0$, for $0 \leq j \leq m$. From the expression for the product fg given by $fg = (\bar{b}_0 + \bar{b}_1 X_1 + \cdots + \bar{b}_t X_t)(a_0 + a_1 X_1 + \cdots + a_m X_m) = \sum_{l=0}^{k+t} \left(\sum_{i+j=l} \bar{b}_i X_i a_j Y_j \right)$, and using the relations established in (3.1), we can find explicitly the coefficients for every term of fg (considering a total order as in Definition 2.5 for the products of the elements X_i , Y_j , for instance, $X_k \succ \cdots \succ X_1$ and $Y_t \succ \cdots \succ Y_1$). Now, by assumption, M is a flat R -module, so there exists an R -module homomorphism $\beta : F \rightarrow K$ which fixes the coefficients of every term of the product fg . Consider the elements $w_i := \beta(b_i) - b_i$, $i = 1, \dots, k$ in F . Then the element $h = w_0 + w_1 X_1 + \cdots + w_k X_k$ is an element of $F[X]$ which satisfies that $hg = 0$. Note that F is skew-Armendariz by Proposition, because R is skew-Armendariz and F is a free R -module. Hence, $w_0 a_j = 0$, for $j = 1, \dots, k$, and so $b_0 a_j \in K$, for every j , that is, $\bar{b}_0 a_j = 0$ in M , which shows that M is skew-Armendariz. \square

For the next theorem, consider the set $\text{Ann}_B(2^{M_B}) := \{\text{ann}_B(U) \mid U \subseteq M_B\}$, where M_B is an B -module. Theorem 3.13 generalizes [2], Theorem 2.24, from Ore extensions to skew PBW extensions.

Theorem 3.13. *If A is a σ -PBW extension of a ring R , and M_R is a (Σ, Δ) -compatible right R -module, then the following conditions are equivalent:*

- (1) M_R is a skew-Armendariz module;
- (2) The map $\psi : \text{Ann}_R(2^{M_R}) \rightarrow \text{Ann}_S(2^{M(X)_A})$, defined by $C \rightarrow CA$, for all $C \in \text{Ann}_R(2^{M_R})$, is bijective.

Proof. (1) \Rightarrow (2) First of all, let us see that $\text{ann}_R(U)A = \text{ann}_A(U)$, for every $U \subseteq M_R$. If $f \in \text{ann}_R(U)A$, then f is expressed as $f = r_0 + r_1Y_1 + \cdots + r_kY_p$, where $r_i \in \text{ann}_R(U)$ and $Y_i \in \text{Mon}(A)$, for every i . If $g = m_0 + m_1X_1 + \cdots + m_kX_k \in U$, then $gf = 0$ because $gr_i = 0$, for $1 \leq i \leq p$, so $gf = 0$, i.e., $f \in \text{ann}_A(U)$. Now, if $h = a_0 + a_1Y_1 + \cdots + a_tY_t \in \text{ann}_A(U)$, then $gh = 0$, for every $g = m_0 + m_1X_1 + \cdots + m_kX_k \in U$. Since M_R is a skew-Armendariz module, we have $m_0a_j = 0$, for $j = 0, \dots, t$, and by the (Σ, Δ) -compatibility of M_R , we can see that $ga_j = 0$, for every j , so $h \in \text{ann}_R(U)A$.

The application $\psi : \{\text{ann}_R(U) \mid U \subseteq M_R\} \rightarrow \{\text{ann}_A(U) \mid U \subseteq M\langle X \rangle_A\}$, defined by $C \rightarrow CA$, for every $C \in \{\text{ann}_R(U) \mid U \subseteq M_R\}$, is well defined, since $\text{ann}_R(U)A = \text{ann}_A(U)$, for every $U \subseteq M_R$. From the (Σ, Δ) -compatibility of M_R , we can deduce that $\text{ann}_A(V) \cap R = \text{ann}_R(V_0)$, for every $V \subseteq M\langle X \rangle_A$, where $V_0 \subseteq M$ is the set of coefficients of V . This fact guarantees that the map $\psi' : \{\text{ann}_A(U) \mid U \subseteq M\langle X \rangle_A\} \rightarrow \{\text{ann}_R(U) \mid U \subseteq M_R\}$, defined by $D \rightarrow D \cap R$, for every $D \in \{\text{ann}_A(U) \mid U \subseteq M\langle X \rangle_A\}$. Note that $\psi'\psi = \text{id}$, so ψ is an injective map. Consider the set $B \in \{\text{ann}_A(U) \mid U \subseteq M\langle X \rangle_A\}$. Then $B = \text{ann}_A(J)$, for some $J \subseteq M\langle X \rangle_A$. Let B_1 and J_1 be the set of coefficients of elements of B and J , respectively. The aim is to prove that $\text{ann}_R(J_1) = B_1R$. With this in mind, let $m = m_0 + m_1X_1 + \cdots + m_kX_k \in J$ and $f = a_0 + a_1Y_1 + \cdots + a_tY_t \in B$. Then $mf = 0$, and using both assumptions on M_R , it follows that $m_ib_j = 0$, for every m_i and b_j , whence $J_1B_1 = 0$, that is, $B_1R \subseteq \text{ann}_R(J_1)$. Note that due to the (Σ, Δ) -compatibility of M_R , we can assert that $\text{ann}_R(J_1) \subseteq B_1R$, which shows that $\text{ann}_R(J_1) = B_1R$, and so $\text{ann}_A(J) = B_1RA$. This proves that ψ is a surjective function.

(2) \Rightarrow (1) Consider the elements $m = m_0 + m_1X_1 + \cdots + m_kX_k \in M\langle X \rangle_A$ and $f \in a_0 + a_1Y_1 + \cdots + a_tY_t \in A$ with $mf = 0$. It is clear that $f \in \text{ann}_A(\{m\}) = \text{ann}_R(U)A$, where $U \subseteq M_R$. In this way, the elements b_0, \dots, b_n belong to $\text{ann}_R(U)$, whence $mb_j = 0$, for every j . Therefore, $m_0b_j = 0$, for $0 \leq j \leq t$, which concludes the proof. \square

The following theorem generalizes [2], Theorem 2.25.

Theorem 3.14. *If A is a σ -PBW extension of a ring R , and M_R is a linearly skew-Armendariz module with $R \subseteq M$, then for every idempotent elements e of R , we have $\sigma_i(e) = e$ and $\delta_i(e) = 0$, for every $i = 1, \dots, n$.*

Proof. Note that M_R is a linearly skew-Armendariz module with $R \subseteq M_R$, so R_R is also linearly skew-Armendariz. Let us prove the assertion for R_R .

Consider an idempotent element e of R . Then $\delta_i(e) = \sigma_i(e)\delta_i(e) + \delta_i(e)e$. Let $f, g \in A$ given by $f = \delta_i(e) + 0x_1 + \cdots + 0x_{i-1} + \sigma_i(e)x_i + 0x_{i+1} + \cdots + 0x_n$, and $g = e - 1 + (e - 1)x_1 + \cdots + (e - 1)x_n$, respectively. Recall that $\delta_i(1) = 0$, for every i . Let us show that $fg = 0$:

$$\begin{aligned} fg &= \delta_i(e)(e - 1) + \left(\sum_{j=1}^n \delta_i(e)(e - 1)x_j \right) + \sigma_i(e)x_i(e - 1) + \sum_{j=1}^n \sigma_i(e)x_i(e - 1)x_j \\ &= \delta_i(e)(e - 1) + \left(\sum_{j=1}^n \delta_i(e)(e - 1)x_j \right) + \sigma_i(e)[\sigma_i(e - 1)x_i + \delta_i(e - 1)] \\ &\quad + \sum_{j=1}^n \sigma_i(e)[\sigma_i(e - 1)x_i + \delta_i(e - 1)]x_j. \end{aligned}$$

Equivalently,

$$\begin{aligned}
fg &= \delta_i(e)(e-1) + \left(\sum_{j=1}^n \delta_i(e)(e-1)x_j \right) + \sigma_i(e)[(\sigma_i(e) - \sigma_i(1))x_i + \delta_i(e)] \\
&+ \sum_{j=1}^n \sigma_i(e)[(\sigma_i(e) - \sigma_i(1))x_i + \delta_i(e)]x_j \\
&= \delta_i(e)(e-1) + \left(\sum_{j=1}^n \delta_i(e)(e-1)x_j \right) + \sigma_i(e)[\sigma_i(e)x_i - x_i + \delta_i(e)] \\
&+ \sum_{j=1}^n \sigma_i(e)[\sigma_i(e)x_i - x_i + \delta_i(e)]x_j \\
&= \delta_i(e)e - \delta_i(e) + \left(\sum_{j=1}^n (\delta_i(e)e - \delta_i(e))x_j \right) + \sigma_i(e)x_i - \sigma_i(e)x_i + \sigma_i(e)\delta_i(e) \\
&+ \sum_{j=1}^n (\sigma_i(e)x_i - \sigma_i(e)x_i + \sigma_i(e)\delta_i(e))x_j \\
&= \delta_i(e)e - \delta_i(e) + \sum_{j=1}^n \delta_i(e)ex_j - \sum_{j=1}^n \delta_i(e)x_j + \sigma_i(e)\delta_i(e) + \sum_{j=1}^n \sigma_i(e)\delta_i(e)x_j = 0.
\end{aligned}$$

From Definition 3.3 we obtain $\delta_i(e)(e-1) = 0$, i.e., $\delta_i(e)e = \delta_i(e)$, and hence $\sigma_i(e)\delta_i(e) = 0$.

Now, consider the elements s and t of A given by $s = \delta_i(e) - (1 - \sigma_i(e))x_i$ and $t = e + \sum_{j=1}^n ex_j$, respectively. Let us show that $st = 0$:

$$\begin{aligned}
st &= \delta_i(e)e + \left(\delta_i(e)e \sum_{j=1}^n x_j \right) - (1 - \sigma_i(e))x_i e - \left((1 - \sigma_i(e))x_i e \sum_{j=1}^n x_j \right) \\
&= \delta_i(e)e + \left(\delta_i(e)e \sum_{j=1}^n x_j \right) - x_i e + \sigma_i(e)x_i e - x_i e \sum_{j=1}^n x_j + \sigma_i(e)x_i e \sum_{j=1}^n x_j \\
&= \delta_i(e)e + \left(\delta_i(e)e \sum_{j=1}^n x_j \right) - (\sigma_i(e)x_i + \delta_i(e)) + \sigma_i(e)(\sigma_i(e)x_i + \delta_i(e)) \\
&- \left((\sigma_i(e)x_i + \delta_i(e)) \sum_{j=1}^n x_j \right) + \sigma_i(e)(\sigma_i(e)x_i + \delta_i(e)) \sum_{j=1}^n x_j \\
&= \delta_i(e)e + \left(\delta_i(e)e \sum_{j=1}^n x_j \right) - \sigma_i(e)x_i - \delta_i(e) + \sigma_i(e)x_i + \sigma_i(e)\delta_i(e) - \sigma_i(e)x_i \sum_{j=1}^n x_j \\
&- \delta_i(e) \sum_{j=1}^n x_j + \sigma_i(e)x_i \sum_{j=1}^n x_j + \sigma_i(e)\delta_i(e) \sum_{j=1}^n x_j.
\end{aligned}$$

Since $\delta_i(e) = \delta_i(e)e$ and $\sigma_i(e)\delta_i(e) = 0$, then $st = 0$. By Armendariz condition we know that $\delta_i(e)e = 0$, which shows that $\delta_i(e) = 0$.

Consider the elements $u, v \in A$ given by $u = 1 - e + (1 - e)\sigma_i(e)x_i$ and $v = e + (e - 1)\sigma_i(e)x_i$. We have the equalities

$$\begin{aligned}
uv &= e + (e - 1)\sigma_i(e)x_i - e^2 - e(e - 1)\sigma_i(e)x_i + (1 - e)\sigma_i(e)x_i e + (1 - e)\sigma_i(e)x_i(e - 1)\sigma_i(e)x_i \\
&= e\sigma_i(e)x_i - \sigma_i(e)x_i - e\sigma_i(e)x_i + e\sigma_i(e)x_i + (1 - e)\sigma_i(e)(\sigma_i(e)x_i + \delta_i(e)) \\
&\quad + (1 - e)\sigma_i(e)(\sigma_i(e)x_i - x_i + \delta_i(e))\sigma_i(e)x_i \\
&= -\sigma_i(e)x_i + e\sigma_i(e)x_i + \sigma_i(e)x_i + \sigma_i(e)\delta_i(e) - e\sigma_i(e)x_i - e\sigma_i(e)\delta_i(e) \\
&\quad + [\sigma_i(e)x_i - \sigma_i(e)x_i + \sigma_i(e)\delta_i(e) - e\sigma_i(e)x_i + e\sigma_i(e)x_i - e\sigma_i(e)\delta_i(e)]\sigma_i(e)x_i = 0.
\end{aligned}$$

Using that $\delta_i(e) = 0$, we obtain $(1 - e)(e - 1)\sigma_i(e) = 0$, i.e., $e\sigma_i(e) = \sigma_i(e)$.

Finally, let $w = e + e(1 - \sigma_i(e))x_i$, $z = 1 - e - e(1 - \sigma_i(e))x_i$ be elements of A . Then

$$\begin{aligned}
wz &= e - e^2 - e^2(1 - \sigma_i(e))x_i + e(1 - \sigma_i(e))x_i - e(1 - \sigma_i(e))x_i e - e(1 - \sigma_i(e))x_i e(1 - \sigma_i(e))x_i \\
&= -e(1 - \sigma_i(e))(\sigma_i(e)x_i + \delta_i(e)) - e(1 - \sigma_i(e))[\sigma_i(e(1 - \sigma_i(e)))x_i + \delta_i(e(1 - \sigma_i(e))))x_i.
\end{aligned}$$

Using that $\delta_i(e) = 0$ and $e\sigma_i(e) = \sigma_i(e)$, we can see that $wz = 0$. Hence, $e(-e(1 - \sigma_i(e))) = 0$, which shows that $e\sigma_i(e) = e$, and so $\sigma_i(e) = e$. \square

In [1], Agayev et. al., introduced the notion of abelian module: a right B -module M_B is called *abelian*, if for any elements $m \in M$, $r \in B$, and every idempotent $e \in B$, we have $mae = mea$. They proved that every Armendariz module, and hence every reduced module is abelian. The next theorem generalizes [2], Theorem 2.26, from Ore Extensions to skew PBW extensions.

Theorem 3.15. *If A is a σ -PBW extension of a ring R , and M_R is a linearly skew-Armendariz module with $R \subseteq M$, then M_R is an abelian module.*

Proof. Consider M_R a linearly skew-Armendariz module, the elements defined by $m_1 = me - \sum_{i=1}^n mer(1 - e)x_i$, $m_2 = m(1 - e) - \sum_{i=1}^n m(1 - e)rex_i \in M\langle X \rangle_A$, and $f_1 = (1 - e) + \sum_{i=1}^n er(1 - e)x_i$, $f_2 = e + \sum_{i=1}^n (1 - e)rex_i$, where $e \in R$ is an idempotent element, and $r \in R, m \in M$. Let us show that $m_1 f_1 = m_2 f_2 = 0$. Recall that $\sigma_i(e) = e$ and $\delta_i(e) = 0$, for $i = 1, \dots, n$ (Theorem 3.14), so $x_i e = e x_i$, for every i . We have the following equalities:

$$\begin{aligned}
m_1 f_1 &= \left(me - \sum_{i=1}^n mer(1 - e)x_i \right) \left((1 - e) + \sum_{i=1}^n er(1 - e)x_i \right) \\
&= \left(me - \sum_{i=1}^n merx_i + \sum_{i=1}^n merex_i \right) \left(1 - e + \sum_{i=1}^n erx_i - \sum_{i=1}^n erex_i \right) \\
&= me - me^2 + \sum_{i=1}^n me^2 rx_i - \sum_{i=1}^n me^2 rex_i - \sum_{i=1}^n merx_i + \sum_{i=1}^n merx_i e \\
&\quad - \left(\sum_{i=1}^n merx_i \right) \left(\sum_{i=1}^n erx_i \right) + \left(\sum_{i=1}^n merx_i \right) \left(\sum_{i=1}^n erex_i \right) - \sum_{i=1}^n merex_i + \sum_{i=1}^n merex_i e \\
&\quad + \left(\sum_{i=1}^n merex_i \right) \left(\sum_{i=1}^n erx_i \right) - \left(\sum_{i=1}^n merex_i \right) \left(\sum_{i=1}^n erex_i \right),
\end{aligned}$$

or equivalently,

$$\begin{aligned} m_1 f_1 = & - \left(\sum_{i=1}^n merex_i r \right) \left(\sum_{i=1}^n x_i \right) + \left(\sum_{i=1}^n merex_i r \right) \left(\sum_{i=1}^n ex_i \right) \\ & + \left(\sum_{i=1}^n mere^2 x_i r \right) \left(\sum_{i=1}^n x_i \right) - \left(\sum_{i=1}^n mere^2 x_i r \right) \left(\sum_{i=1}^n ex_i \right) = 0, \end{aligned}$$

and,

$$\begin{aligned} m_2 f_2 = & \left(m(1-e) - \sum_{i=1}^n m(1-e)rex_i \right) \left(e + \sum_{i=1}^n (1-e)rex_i \right) \\ = & \left(m - me - \sum_{i=1}^n mrex_i + \sum_{i=1}^n merex_i \right) \left(e + \sum_{i=1}^n rex_i - \sum_{i=1}^n erex_i \right) \\ = & me + \sum_{i=1}^n mrex_i - \sum_{i=1}^n merex_i - me^2 - \sum_{i=1}^n merex_i + \sum_{i=1}^n me^2 rex_i \\ & - \sum_{i=1}^n mrex_i e - \left(\sum_{i=1}^n mrex_i \right) \left(\sum_{i=1}^n rex_i \right) + \left(\sum_{i=1}^n mrex_i \right) \left(\sum_{i=1}^n erex_i \right) \\ & + \sum_{i=1}^n merex_i e + \left(\sum_{i=1}^n merex_i \right) \left(\sum_{i=1}^n rex_i \right) - \left(\sum_{i=1}^n merex_i \right) \left(\sum_{i=1}^n erex_i \right) \\ = & - \left(\sum_{i=1}^n mrex_i \right) \left(\sum_{i=1}^n rex_i \right) + \left(\sum_{i=1}^n mrex_i e \right) \left(\sum_{i=1}^n rex_i \right) \\ & + \left(\sum_{i=1}^n merex_i \right) \left(\sum_{i=1}^n rex_i \right) - \left(\sum_{i=1}^n merex_i e \right) \left(\sum_{i=1}^n rex_i \right) = 0. \end{aligned}$$

By assumption, M_R is linearly skew-Armendariz, so $meer(1-e) = 0$ and $m(1-e)(1-e)re$, or what is the same, $mer = mere$ and $mre = mere$, respectively, whence $mer = mre$, i.e., M_R is an abelian module. \square

Corollary 3.16. *If A is a σ -PBW extension of a ring R , and M_R is a skew-Armendariz module with $R \subseteq M$, then M_R is an abelian module.*

Proof. The assertion follows from Theorem 3.15 and the fact that every skew-Armendariz module is linearly skew-Armendariz module. \square

Note that [2], Corollary 2.27 is a particular case of Corollary 3.16. Next theorem generalizes [2], Theorem 2.28.

Theorem 3.17. *If M_R is a reduced module, then M_R is a p.p.-module if and only if M_R is a p.q.-Baer module.*

Proof. Consider M_R a reduced right R -module. From Proposition 3.6, we know that for every elements m of M and r of R , the equality $mr = 0$ implies $mRr = 0$, which means that $\text{ann}_R(\{m\}) = \text{ann}_R(mR)$, and so, $\text{ann}_R(\{m\}) = \text{ann}_R(mR)$. \square

Theorem 3.18 generalizes [2], Theorem 2.29.

Theorem 3.18. *If A is a σ -PBW extension of a ring R , and M_R is an (Σ, Δ) -compatible skew-Armendariz right R -module with $R \subseteq M$, then M_R is p.p.-module if and only if $M\langle X \rangle_A$ is p.p.-module*

Proof. Let M_R be a p.p.-module. Consider an element m of $M\langle X \rangle$ given by the expression $m = m_0 + m_1X_1 + \cdots + m_kX_k$. We know that $\text{ann}_R(\{m_i\}) = e_iR$, for idempotent elements $e_i \in R$, for every i . Consider the product of the elements e 's, that is, let $e := e_0e_1 \cdots e_k$. Note that e is idempotent (M_R is abelian by Corollary 3.16). Therefore we have $eR = \bigcap_{i=0}^k \text{ann}_R(\{m_i\})$. From Theorem 3.14 we know that $\sigma_i(e) = e$ and $\delta_i(e) = 0$, for every i , which guarantees that the product me is zero, and so, $eA \subseteq \text{ann}_A(\{m\})$. Now, if $g \in \text{ann}_A(\{m\})$ is given by $g = b_0 + b_1X_1 + \cdots + b_tX_t$, then $m_0b_j = 0$ for $0 \leq j \leq t$ (M_R is skew-Armendariz). In this way, $b_0 \in eR$, whence $g \in eA$, that is, $\text{ann}_A(\{m\}) = eA$. In other words, we have shown that $M\langle X \rangle$ is a p.p.-module over A .

Conversely, if $M\langle X \rangle_A$ is a p.p.-module and m is an element of M , for an idempotent element $e \in A$ given by $e = e_0 + e_1X_1 + \cdots + e_pX_p$, we have $e(1 - e) = 0 = (1 - e)e$, or equivalently, $(e_0 + e_1X_1 + \cdots + e_pX_p)(1 - e_0 - e_1X_1 - \cdots - e_pX_p) = 0 = (1 - e_0 - e_1X_1 - \cdots - e_pX_p)(e_0 + e_1X_1 + \cdots + e_pX_p)$. As we know, M_R is skew-Armendariz, so $e_0(1 - e_0) = 0$ and $(1 - e_0)e_i = 0$, for every i , which means that $e_0e_i = 0$, $e_i = e_0e_i$, that is, $e_i = 0$. Then $e = e_0^2 = e_0 \in R$, and $\text{ann}_A(\{m\}) = eA$, whence $\text{ann}_R(\{m\}) = eR$, i.e., M_R is a p.p.-module. \square

Theorem 3.19 extends [2], Theorem 2.30, from Ore extensions to σ -PBW extensions.

Theorem 3.19. *If A is a σ -PBW extension of a ring R , M_R is an (Σ, Δ) -compatible skew-Armendariz module with $R \subseteq M$, then M_R is Baer if and only if $M\langle X \rangle_A$ is Baer.*

Proof. Suppose that M_R is a Baer module. Let $J \subseteq M\langle X \rangle$ and J_0 the set of elements m of M such that m is the leading coefficient of some non-zero element of J . Using that M_R is Baer, there exists $e^2 = e \in R$ with $\text{ann}_R(J_0) = eR$, and hence $eA \subseteq \text{ann}_A(J)$, by Proposition 3.7. Now, consider an element $g = b_0 + b_1X_1 + \cdots + b_tX_t \in \text{ann}_A(J)$. Since M_R is skew-Armendariz, $J_0b_j = 0$, for $0 \leq j \leq t$. This fact means that $b_j = eb_j$, for every j , and $g = eg \in A$, so $\text{ann}_A(J) = eA$ and $M\langle X \rangle_A$ is a Baer module. Finally, if $M\langle X \rangle_A$ is a Baer module y $C \subseteq M$, then $C[X] \subseteq M\langle X \rangle$, and since $M\langle X \rangle$ is Baer, there exists an idempotent element $e = e_0 + e_1X_1 + \cdots + e_pX_p \in A$ with $\text{ann}_A[C[X]] = eA$. In this way, $Ce_0 = \{0\}$ and $e_0R \subseteq \text{ann}_R(C)$. On the other hand, if $r \in \text{ann}_R(C)$, then $C[X]r = 0$ (Proposition 3.8), whence $t = et$, that is, $t = e_0t \in e_0R$, which proves that $\text{ann}_R(C) = e_0R$, and hence M_R is a Baer module. \square

The next proposition generalizes [2], Proposition 2.32, Corollaries 2.33 and 2.34.

Proposition 3.20. *Let A be a σ -PBW extension of a ring R , and let M_R be an (Σ, Δ) -compatible and reduced module. If m is a torsion element in $M\langle X \rangle$, i.e., $mh = 0$, for some non-zero element $h \in A$, then there exists a non-zero element $c \in R$ such that $mc = 0$.*

Proof. Consider the elements $m = m_0 + m_1X_1 + \cdots + m_kX_k \in M\langle X \rangle$, $0 \neq f = b_0 + b_1Y_1 + \cdots + b_tY_t \in A$, $a_t \neq 0$, with $mf = 0$. We have $mf = (m_0 + m_1X_1 + \cdots + m_kX_k)(b_0 + b_1Y_1 + \cdots + b_tY_t) = \sum_{l=0}^{k+t} \left(\sum_{i+j=l} m_iX_i b_jY_j \right)$. Note that $\text{lc}(mf) = m_k\sigma^{\alpha_k}(b_t)c_{\alpha_k, \beta_t} = 0$. Since A is

bijjective, $m_k \sigma^{\alpha_k}(b_t) = 0$, and by the Σ -compatibility of M_R , $m_k b_t = 0$. The idea is to prove that $m_p b_q = 0$ for $p + q \geq 0$. We proceed by induction. Suppose that $m_p b_q = 0$ for $p + q = k + t, k + t - 1, k + t - 2, \dots, l + 1$ for some $l > 0$. By Proposition 3.6 and expression (3.1), we obtain $m_p X_p b_q Y_q = 0$, for these values of $p + q$. In this way, we only consider the sum of the products $m_u X_u b_v Y_v$, where $u + v = l, l - 1, l - 2, \dots, 0$. Fix u and v . Consider the sum of all terms of mf having exponent $\alpha_u + \beta_v$. From expression (3.1), Proposition 3.6, and the assumption $mf = 0$, we know that the sum of all coefficients of all these terms can be written as

$$m_u \sigma^{\alpha_u}(b_v) c_{\alpha_u, \beta_v} + \sum_{\alpha_{u'} + \beta_{v'} = \alpha_u + \beta_v} m_{u'} \sigma^{\alpha_{u'}}(\sigma's \text{ and } \delta's \text{ evaluated in } b_{v'}) c_{\alpha_{u'}, \beta_{v'}} = 0. \quad (3.5)$$

By assumption, we know that $m_p b_q = 0$, for $p + q = k + t, k + t - 1, \dots, l + 1$. So, Proposition 3.6 guarantees that the product

$$m_p(\sigma's \text{ and } \delta's \text{ evaluated in } b_q) \quad (\text{any order of } \sigma's \text{ and } \delta's)$$

is equal to zero. Then $[(\sigma's \text{ and } \delta's \text{ evaluated in } b_q) a_p]^2 = 0$, and hence we obtain the equality $(\sigma's \text{ and } \delta's \text{ evaluated in } b_q) m_p = 0$ (M_R is reduced). In this way, multiplying (3.5) by m_l , and using the fact that the elements $c_{i,j}$ in Definition 2.1 (iv) are in the center of R ,

$$m_u \sigma^{\alpha_u}(b_v) a_k c_{\alpha_u, \beta_v} + \sum_{\alpha_{u'} + \beta_{v'} = \alpha_u + \beta_v} m_{u'} \sigma^{\alpha_{u'}}(\sigma's \text{ and } \delta's \text{ evaluated in } b_{v'}) m_l c_{\alpha_{u'}, \beta_{v'}} = 0, \quad (3.6)$$

whence, $m_u \sigma^{\alpha_u}(b_0) a_l = 0$. Since $u + v = l$ and $v = 0$, then $u = l$, so $m_l \sigma^{\alpha_l}(b_0) m_l = 0$, i.e., $[m_l \sigma^{\alpha_l}(b_0)]^2 = 0$, from which $m_l \sigma^{\alpha_l}(b_0) = 0$ and $m_l b_0 = 0$, by Proposition 3.6. Therefore, we now have to study the expression (3.5) for $0 \leq u \leq l - 1$ and $u + v = l$. If we multiply (3.6) by m_{l-1} , we obtain

$$m_u \sigma^{\alpha_u}(b_v) m_{l-1} c_{\alpha_u, \beta_v} + \sum_{\alpha_{u'} + \beta_{v'} = \alpha_u + \beta_v} m_{u'} \sigma^{\alpha_{u'}}(\sigma's \text{ and } \delta's \text{ evaluated in } b_{v'}) m_{k-1} c_{\alpha_{u'}, \beta_{v'}} = 0.$$

Using a similar reasoning as above, we can see that $m_u \sigma^{\alpha_u}(b_1) m_{l-1} c_{\alpha_u, \beta_1} = 0$. Since A is bijective, $m_u \sigma^{\alpha_u}(b_1) m_{l-1} = 0$, and using the fact $u = l - 1$, we have $[m_{l-1} \sigma^{\alpha_{l-1}}(b_1)] = 0$, which imply $m_{l-1} \sigma^{\alpha_{l-1}}(b_1) = 0$, that is, $m_{l-1} b_1 = 0$. Continuing in this way we prove that $m_i b_j = 0$, for $i + j = l$. Therefore $a_i b_j = 0$, for $0 \leq i \leq m$ and $0 \leq j \leq t$. Finally, since f is a non-zero element of A , we may consider $c := a_t \neq 0$, and hence $mc = 0$, by Proposition 3.8. \square

3.2 Skew quasi-Armendariz modules over σ -PBW extensions

In [17], Hirano called a ring B *quasi-Armendariz*, if whenever $f(x)B[x]g(x) = 0$, where $f(x) = a_0 + a_1x + \dots + a_mx^m$ and $g(x) = b_0 + b_1x + \dots + b_nx^n \in B[x]$, then $a_i B b_j = 0$, for all i and j . In the same paper, Hirano called a right B -module *quasi-Armendariz*, if whenever $m(x)B[x]f(x) = 0$, where $m(x) = m_0 + m_1x + \dots + m_sx^s \in M\langle X \rangle$ and $f(x) = a_0 + a_1x + \dots + a_tx^t \in B[x]$, implies that $m_i B a_j = 0$, for all i, j . Now, in [2], Alhevaz and Moussavi introduced the notion of *skew quasi-Armendariz* for modules in the following way: let M_B be a right B -module, α an endomorphism of B and δ an α -derivation of B . M_B is called *skew quasi-Armendariz*, if whenever $m(x) = m_0 + m_1x + \dots + m_kx^k \in M\langle X \rangle$, $f(x) = b_0 + b_1x + \dots + b_nx^n \in B[x; \alpha, \delta]$ satisfy $m(x)B[x; \alpha, \delta]f(x) = 0$, we have $m_i x^i B x^t b_j x^j = 0$ for $t \geq 0$, $i = 0, 1, \dots, k$ and $j = 0, 1, \dots, n$. With the aim of extending these definitions for the class of σ -PBW extensions, we present the following definition:

Definition 3.21. If A is a σ -PBW extension of a ring R , and M_R is a right R -module, M_R is called *skew quasi-Armendariz*, if whenever $m = m_0 + m_1X_1 + \cdots + m_kX_k \in M\langle X \rangle$, $f = a_0 + a_1Y_1 + \cdots + a_pY_p \in A$ with $mAf = 0$, then $m_iX_iRX_ta_jY_j = 0$, for any $X_t \in \text{Mon}(A)$, and any values of i, j .

The next theorem generalizes [2], Theorem 3.2, for the Ore extensions to σ -PBW extensions, and its proof is easy.

Theorem 3.22. *If A is a quasi-commutative σ -PBW extension of R , and M_R is an Σ -compatible right R -module, then:*

(1) *The following assertions are equivalent:*

- (a) *for every $m \in M\langle X \rangle_A$, $(\text{ann}_A\{mA\} \cap R)\langle x_1, \dots, x_n \rangle = \text{ann}_A\{mA\}$.*
- (b) *for every $m = m_0 + m_1X_1 + \cdots + m_kX_k \in M\langle X \rangle_A$ and $f = a_0 + a_1Y_1 + \cdots + a_tY_t \in A$, the equality $mAf = 0$ implies $m_iRa_j = 0$, for every i, j .*

(2) *If M_R is a skew quasi-Armendariz module and m is an element of $M\langle X \rangle_A$, the equality $\text{ann}_A\{mA\} \neq \{0\}$ implies that $\text{ann}_A\{mA\} \cap R \neq \{0\}$.*

Theorem 3.23 generalizes [2], Theorem 3.3.

Theorem 3.23. *If A is a quasicommutative σ -PBW extension of R , and M_R is an Σ -compatible right R -module, then the following assertions are equivalent:*

- (1) *M_R is a skew quasi-Armendariz module;*
- (2) *The map $\psi : \text{Ann}_R(\text{sub}(M_R)) \rightarrow \text{Ann}_A(\text{sub}(M\langle X \rangle_A))$, defined by $\psi(\text{ann}_R(M)) = \text{ann}_A(N) = \text{ann}_A(N[X])$, for all $N \in \text{sub}(M_R)$, is bijective, where $\text{sub}(M_R)$ and $\text{sub}(M\langle X \rangle_A)$ denote the sets of submodules of M_R and $M\langle X \rangle_A$, respectively.*

Proof. The proof follows from [46], Theorem 4.12. □

Theorem 3.24 generalizes [2], Theorem 3.6, Corollary 3.7 and [17], Corollary 3.3.

Theorem 3.24. *If A is a σ -PBW extension of R , and M_R is a skew quasi-Armendariz right R -module which is (Σ, Δ) -compatible, then M_R satisfies the ascending chain condition on annihilators of submodules if and only if so does $M\langle X \rangle_A$.*

Proof. Suppose that M_R satisfies the ascending chain condition on annihilators of submodules. Consider the chain of annihilator of submodules of $M\langle X \rangle_A$ given by $I_1 \subseteq I_2 \subseteq \cdots$. Then, there exist submodules K_i of $M\langle X \rangle_A$ with $\text{ann}_A(K_i) = I_i$, for $i = 1, 2, \dots$, and satisfying the relations $K_1 \supseteq K_2 \supseteq \cdots$. Consider the sets M_i consisting of all elements of K_i , for every i . By assumption, M is skew-Armendariz, so that M_i is a submodule of M , for every i . Note that $M_i \supseteq M_{i+1}$, whence $\text{ann}_R(M_1) \subseteq \text{ann}_R(M_2) \subseteq \cdots$. Since we suppose that M_R satisfies the ascending chain condition on annihilators of submodules, then there exists $p \geq 1$ such that $\text{ann}_R(M_i) \subseteq \text{ann}_R(M_p)$, for every value $i \geq p$. Our objective is to show that $\text{ann}_A(K_i) = \text{ann}_A(K_p)$, for these values $i \geq p$. With this in mind, consider an element $f \in \text{ann}_A(K_i)$ given by the expression $f = a_0 + a_1X_1 + \cdots + a_mX_m$. Since M is skew quasi-Armendariz, we have $M_ia_j = 0$, for $0 \leq j \leq m$. This fact implies that

$M_p a_j = 0$, for $0 \leq j \leq m$, and hence Proposition 3.8 guarantees that $K_p f = 0$. Therefore we conclude that $\text{ann}_A(K_i) = \text{ann}_A(K_p)$, for $i \geq p$, i.e., $M\langle X \rangle_A$ satisfies the ascending chain condition on annihilator of submodules.

Conversely, suppose that $M\langle X \rangle_A$ satisfies the ascending chain condition on annihilator of submodules, and consider the chain of annihilator of submodules of M_R given by $J_1 \subseteq J_2 \subseteq \dots$. Then, there exist submodules M_i of M with $\text{ann}_R(M_i) = J_i$, for $i = 1, 2, \dots$, and $M_1 \supseteq M_2 \supseteq \dots$. Note that $M_i[X]$ is a submodule of $M\langle X \rangle$, $M_i[X] \supseteq M_{i+1}[X]$, and, $\text{ann}_A(M_i[X]) \subseteq \text{ann}_A(M_{i+1}[X])$, for every value of i . Since $M\langle X \rangle_A$ satisfies the ascending chain condition on annihilator of submodules, there exists $q \geq 1$ with $\text{ann}_A(M_i[X]) = \text{ann}_A(M_q[X])$, for $i \geq q$. Using that M is skew-Armendariz, it follows that $\text{ann}_R(M_i) = \text{ann}_R(M_q)$, for $i \geq q$. \square

The next theorem generalizes [2], Theorem 3.9, [16], Corollary 2.8, [11], Corollary 2.8, and [18], Theorems 12 and 15.

Theorem 3.25. *If A is a σ -PBW extension of R and M_R is an (Σ, Δ) -compatible right R -module, then M_R is quasi-Baer (respectively, $p.q$ -Baer) if and only if $M\langle X \rangle_A$ is quasi-Baer (respectively, $p.q$ -Baer). In this case, M_R is skew quasi-Armendariz.*

Proof. Suppose that M_R is a quasi-Baer right R -module. Let us see that M_R is skew quasi-Armendariz. With this aim, consider the product

$$(m_0 + m_1 X_1 + \dots + m_k X_k)A(a_0 + a_1 Y_1 + \dots + a_p Y_p) = 0,$$

where $m_0 + m_1 X_1 + \dots + m_k X_k \in M\langle X \rangle$ and $a_0 + a_1 Y_1 + \dots + a_p Y_p \in A$. In particular, if we only take coefficients in R , we have the expression $(m_0 + m_1 X_1 + \dots + m_k X_k)R(a_0 + a_1 Y_1 + \dots + a_p Y_p) = 0$. Since for any $r \in R$ we have the expressions

$$\begin{aligned} 0 &= (m_0 + m_1 X_1 + \dots + m_k X_k)r(a_0 + a_1 Y_1 + \dots + a_p Y_p) \\ &= m_0 r a_0 + m_0 r a_1 Y_1 + \dots + m_0 r a_p Y_p + m_1 X_1 r a_0 + m_1 X_1 r a_1 Y_1 + \dots + m_1 X_1 r a_p Y_p \\ &\quad + \dots + m_k X_k r a_0 + m_k X_k r a_1 Y_1 + \dots + m_k X_k r a_p Y_p \\ &= m_0 r a_0 + m_0 r a_1 Y_1 + \dots + m_0 r a_p Y_p + m_1 \sigma^{\alpha_1}(r a_0) X_1 + m_1 p_{\alpha_1, r a_0} + m_1 \sigma^{\alpha_1}(r a_1) X_1 Y_1 \\ &\quad + m_1 p_{\alpha_1, r a_1} Y_1 + \dots + m_1 \sigma^{\alpha_1}(r a_p) X_1 Y_p + \dots + m_1 p_{\alpha_1, r a_p} Y_p + \dots + m_k \sigma^{\alpha_k}(r a_0) X_k \\ &\quad + m_k p_{\alpha_k, r a_0} + m_k \sigma^{\alpha_k}(r a_1) X_k Y_1 + m_k p_{\alpha_k, r a_1} Y_1 + \dots + m_k \sigma^{\alpha_k}(r a_p) X_k Y_p + m_k p_{\alpha_k, r a_p} Y_p \\ &= m_0 r a_0 + m_0 r a_1 Y_1 + \dots + m_0 r a_p Y_p + m_1 \sigma^{\alpha_1}(r a_0) X_1 + m_1 p_{\alpha_1, r a_0} \\ &\quad + m_1 \sigma^{\alpha_1}(r a_1) c_{\alpha_1, \beta_1} x^{\alpha_1 + \beta_1} + m_1 \sigma^{\alpha_1}(r a_1) p_{\alpha_1, \beta_1} + m_1 p_{\alpha_1, r a_1} Y_1 \\ &\quad + \dots + m_1 \sigma^{\alpha_1}(r a_p) c_{\alpha_1, \beta_p} x^{\alpha_1 + \beta_p} + m_1 \sigma^{\alpha_1}(r a_p) p_{\alpha_1, \beta_p} + \dots + m_1 p_{\alpha_1, r a_p} Y_p \\ &\quad + \dots + m_k \sigma^{\alpha_k}(r a_0) X_k + m_k p_{\alpha_k, r a_0} + m_k \sigma^{\alpha_k}(r a_1) c_{\alpha_k, \beta_1} x^{\alpha_k + \beta_1} + m_k \sigma^{\alpha_k}(r a_1) p_{\alpha_k, \beta_1} \\ &\quad + m_k p_{\alpha_k, r a_1} Y_1 + \dots + m_k \sigma^{\alpha_k}(r a_p) c_{\alpha_k, \beta_p} x^{\alpha_k + \beta_p} + m_k \sigma^{\alpha_k}(r a_p) p_{\alpha_k, \beta_p} + m_k p_{\alpha_k, r a_p} Y_p, \end{aligned} \tag{3.7}$$

we can see that the leading coefficient of this product is $m_k \sigma^{\alpha_k}(r a_p) c_{\alpha_k, \beta_p} = 0$, whence we obtain $m_k \sigma^{\alpha_k}(r a_p) = 0$ (recall that the elements c_{α_k, β_p} are both invertible), and hence $m_k r a_p = 0$ because

M_R is Σ -compatible. In this way, $a_p \in \text{ann}_R(m_k R)$. Now, since

$$\begin{aligned}
m_k X_k r X_t a_p Y_p &= (m_k \sigma^{\alpha_k}(r) X_k + m_k p_{\alpha_k, r})(\sigma^{\alpha_t}(a_p) X_t Y_p + p_{\alpha_t, a_p} Y_p) \\
&= m_k \sigma^{\alpha_k}(r) X_k \sigma^{\alpha_t}(a_p) X_t X_p + m_k \sigma^{\alpha_k}(r) X_k p_{\alpha_t, a_p} Y_p \\
&\quad + m_k p_{\alpha_k, r} \sigma^{\alpha_t}(a_p) X_t X_p + m_k p_{\alpha_k, r} p_{\alpha_t, a_p} Y_p \\
&= m_k \sigma^{\alpha_k}(r) (\sigma^{\alpha_k}(\sigma^{\alpha_t}(a_p)) X_k + p_{\alpha_k, \sigma^{\alpha_t}(a_p)}) X_t X_p + m_k \sigma^{\alpha_k}(r) X_k p_{\alpha_t, a_p} Y_p \\
&\quad + m_k p_{\alpha_k, r} \sigma^{\alpha_t}(a_p) X_t X_p + m_k p_{\alpha_k, r} p_{\alpha_t, a_p} Y_p \\
&= m_k \sigma^{\alpha_k}(r \sigma^{\alpha_t}(a_p)) X_k X_t X_p + m_k \sigma^{\alpha_k}(r) p_{\alpha_k, \sigma^{\alpha_t}(a_p)} X_t X_p + m_k \sigma^{\alpha_k}(r) X_k p_{\alpha_t, a_p} Y_p \\
&\quad + m_k p_{\alpha_k, r} \sigma^{\alpha_t}(a_p) X_t X_p + m_k p_{\alpha_k, r} p_{\alpha_t, a_p} Y_p
\end{aligned} \tag{3.8}$$

from Proposition 3.7 we obtain that all these expressions are zero, so $m_k X_k r X_t a_p Y_p = 0$. As we know, M_R is quasi-Baer, which means that there exists an element $e_k^2 = e_k \in R$ with $\text{ann}_R(m_k R) = e_k R$, whence $a_p = e_k a_p$. If we replace the element r by re_k in (3.7), we obtain

$$0 = (m_0 + m_1 X_1 + \cdots + m_{k-1} X_{k-1}) re_k (a_0 + a_1 Y_1 + \cdots + a_p Y_p),$$

and using a similar reasoning to the above, we see that

$$\begin{aligned}
0 &= (m_0 + m_1 X_1 + \cdots + m_{k-1} X_{k-1}) re_k (a_0 + a_1 Y_1 + \cdots + a_p Y_p) \\
&= m_0 re_k a_0 + m_0 re_k a_1 Y_1 + \cdots + m_0 re_k a_p Y_p + m_1 \sigma^{\alpha_1}(re_k a_0) X_1 + m_1 p_{\alpha_1, re_k a_0} \\
&\quad + m_1 \sigma^{\alpha_1}(re_k a_1) c_{\alpha_1, \beta_1} x^{\alpha_1 + \beta_1} + m_1 \sigma^{\alpha_1}(re_k a_1) p_{\alpha_1, \beta_1} + m_1 p_{\alpha_1, re_k a_1} Y_1 \\
&\quad + \cdots + m_1 \sigma^{\alpha_1}(re_k a_p) c_{\alpha_1, \beta_p} x^{\alpha_1 + \beta_p} + m_1 \sigma^{\alpha_1}(re_k a_p) p_{\alpha_1, \beta_p} + \cdots + m_1 p_{\alpha_1, re_k a_p} Y_p \\
&\quad + \cdots + m_{k-1} \sigma^{\alpha_{k-1}}(re_k a_0) X_{k-1} + m_{k-1} p_{\alpha_{k-1}, re_k a_0} \\
&\quad + m_{k-1} \sigma^{\alpha_{k-1}}(re_k a_1) c_{\alpha_{k-1}, \beta_1} x^{\alpha_{k-1} + \beta_1} + m_{k-1} \sigma^{\alpha_{k-1}}(re_k a_1) p_{\alpha_{k-1}, \beta_1} \\
&\quad + m_{k-1} p_{\alpha_{k-1}, re_k a_1} Y_1 + \cdots + m_{k-1} \sigma^{\alpha_{k-1}}(re_k a_p) c_{\alpha_{k-1}, \beta_p} x^{\alpha_{k-1} + \beta_p} \\
&\quad + m_{k-1} \sigma^{\alpha_{k-1}}(re_k a_p) p_{\alpha_{k-1}, \beta_p} + m_{k-1} p_{\alpha_{k-1}, re_k a_p} Y_p,
\end{aligned}$$

whence $m_{k-1} \sigma^{\alpha_{k-1}}(re_k a_p) c_{\alpha_{k-1}, \beta_p} = 0 \Rightarrow m_{k-1} \sigma^{\alpha_{k-1}}(re_k a_p) = 0 \Rightarrow m_{k-1} re_k a_p = 0$, which implies that $m_{k-1} R a_p = 0$, i.e., $a_p \in \text{ann}_R(m_{k-1} R)$. Again, using a similar reasoning to the above in expression (3.8), we can see that $m_{k-1} X_{k-1} R X_t a_p X_p = 0$. Therefore we have showed that $a_p \in \text{ann}_R(m_k R) \cap \text{ann}_R(m_{k-1} R)$. Since M_R is quasi-Baer, there exists an idempotent element $s \in R$ with $\text{ann}_R(m_{k-1} R) = sR$, whence $a_p = s a_p$. If we define the element e_{k-1} as $e_{k-1} = e_k s$, then we can see that $e_{k-1} \in \text{ann}_R(m_k R) \cap \text{ann}_R(m_{k-1} R)$. Now, if we replace r by re_{k-1} in (3.7), we obtain the equality

$$0 = (m_0 + m_1 X_1 + \cdots + m_{k-2} X_{k-2}) re_{k-1} (a_0 + a_1 Y_1 + \cdots + a_p Y_p).$$

We can show that the relations $m_{k-2} \sigma^{\alpha_{k-2}}(re_{k-1} a_p) c_{\alpha_{k-2}, \beta_p} = 0 \Rightarrow m_{k-2} \sigma^{\alpha_{k-2}}(re_{k-1} a_p) = 0 \Rightarrow m_{k-2} re_{k-1} a_p = 0$, which implies that $m_{k-2} R a_p = 0$, i.e., $a_p \in \text{ann}_R(m_{k-2} R)$, and hence $m_{k-2} X_{k-2} R X_t a_p X_p = 0$. Continuing in this way, we can prove that $m_i X_i R X_t a_p Y_p = 0$, for $i = 0, \dots, k$, and any $X_t \in \text{Mon}(A)$. Similarly, using the total order on $\text{Mon}(A)$, where $Y_p \succ Y_{p-1} \succ \cdots Y_1 \succ 1$, we can show that $m_i X_i R X_t a_{p-1} Y_{p-1} = m_i X_i R X_t a_{p-2} Y_{p-2} = \cdots = m_i X_i R X_t a_1 Y_1 = m_i X_i R X_t a_0 = 0$, which allows us to conclude that M_R is skew quasi-Armendariz.

Now, we will prove that $M\langle X\rangle_A$ is quasi-Baer. Let J be a A -submodule of $M\langle X\rangle$, and consider the set N as the union of the set of the leading coefficients of non-zero elements of J with the set $\{0\}$. Note that N is a submodule of M . By assumption, M_R is quasi-Baer, so there exists an idempotent element e of R with $\text{ann}_R(N) = eR$, which implies that $eA \subseteq \text{ann}_A(J)$ (Proposition 3.7). With the aim of proving that $eA \supseteq \text{ann}_A(J)$, consider an element $f = a_0 + a_1x_1 + \cdots + a_px_p \in \text{ann}_A(J)$. Since M_R is skew quasi-Armendariz, it follows that $Na_j = 0$, for $0 \leq j \leq p$. Then, $b_j = eb_j$, for every j , and $f = ef \in eA$, which guarantees that $eA \supseteq \text{ann}_A(J)$, and hence, $eA = \text{ann}_A(J)$. Conversely, if $M\langle X\rangle_A$ is quasi-Baer and I is a submodule of M , it follows that $I[X]$ is a submodule of $M\langle X\rangle$, and since $M\langle X\rangle$ is quasi-Baer, there exists $e^2 = e = e_0 + e_1 + \cdots + e_lx_l \in A$ with $\text{ann}_A(I[X]) = eA$. Note that $Ie_0 = 0$ and $e_0R \subseteq \text{ann}_R(I)$. Finally, if $s \in \text{ann}_R(I)$, then $I[X]s = 0$ (Proposition 3.8), and therefore $t = et$, which implies $t = e_0t \in e_0R$, that is, $e_0R \supseteq \text{ann}_R(I)$, i.e., $e_0R = \text{ann}_R(I)$. This concludes the proof. \square

Remark 3.26. • Note that a ring B is right p.q.-Baer if and only if B_B is a p.q.-Baer module. However, this does not hold for the property of being p.q.-Baer. More exactly, there exists a p.q.-Baer right B -module such that B is not right p.q.-Baer ([2], Example 3.10).

• The condition on the (Σ, Δ) -compatibility in Theorem 3.25 can not be dropped, since there exists an example of a ring B such that $B[x; \delta]$ is Baer, and hence quasi-Baer, but B is not quasi-Baer, see [5], Example 1.

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